Exact Factorizations of G-crossed Braided Fusion Categories

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Agenda

- Why study fusion categories?
- What is a category?
- Operations in categories
- Fusion categories
- Gradings
- G-crossed braided fusion categories
- Exact factorization
- Our theorem

Why study fusion categories?

Quantum Physics | **Quantum Computing**

Example:

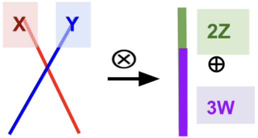
In quantum physics, particles are **anyons** = **strings,** or objects in a category.

Two anyons can fuse into a new anyon.

There's two operations: $\oplus$ and $\otimes$

$X \otimes Y := 2Z \oplus 3W$

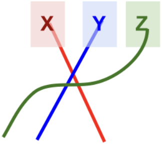
New anyon: 2/5 of Z-anyon and 3/5 of W-anyon

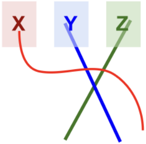


When particles fuse, anyons cross each other, or braid, in a particular order. These are **braidings**.

$Z \otimes (Y \otimes X)$

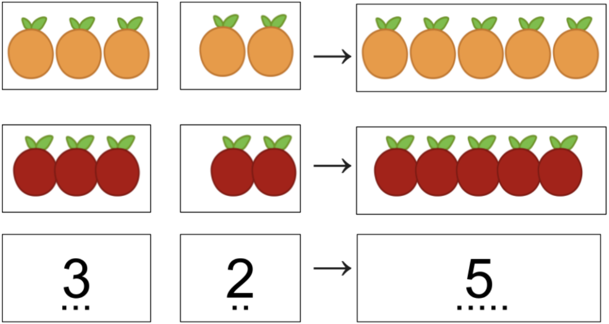
$X \otimes (Z \otimes Y)$





Anyon systems can be represented by **fusion categories**.

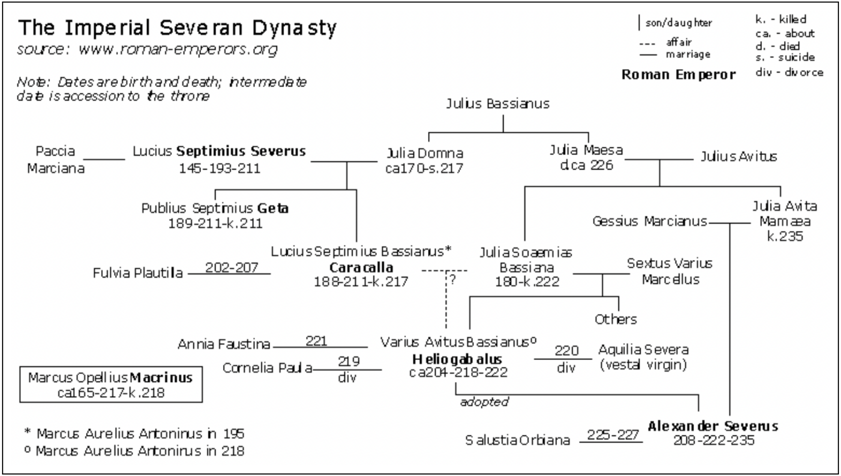
What is abstraction?



It's much easier to understand numbers as abstractions than doing this calculation over and over again..

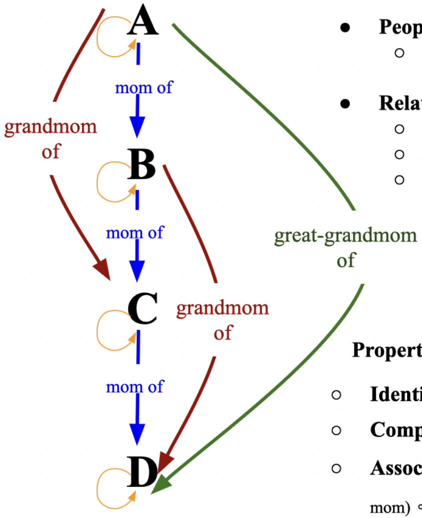
Numbers = abstraction for describing quantities in any set.

Example of a Category: Family Tree



We do not care about the Imperial Severan Dynasty, we care about ALL family trees.

What makes up a family tree?



- **People (Objects)**
  - A, B, C, D
- **Relationships (arrows or morphisms)**
  - Mom of
  - Grandmom of
  - Great-grandmom of

Properties of relationships

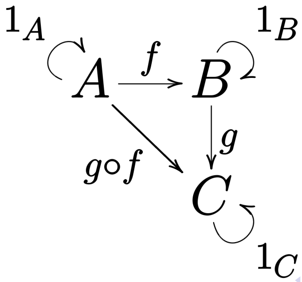
- **Identity**: I am myself
- **Composition**: mom ∘ mom = grandmom
- **Associativity**: mom ∘ (mom ∘ mom) = (mom ∘ mom) ∘ mom

What is a category?

Definition

A **category** $\mathcal{C}$ consists of:

- A set of objects
- For any two objects A and B, a set of arrows (morphisms) between them, $\mathcal{C}(A, B)$, where
  - Associative composition
  - Identity morphism for each object



Operations in Categories

By default, the only operation we have in categories is composition of morphisms. What if we want to add or multiply in categories?

- **Abelian ($\oplus$) category**: objects and morphisms can be *added* with $\oplus$.
  - Object X is **simple** if it has no subobjects except the zero object or itself. (simples = primes)
  - Category is **semisimple** if every object is the direct sum of simple objects. (semisimple = prime factorization exists)
- **Monoidal ($\otimes$) category**: category with a multiplication operation $\otimes$
  - Associativity condition on $\otimes$
- A **fusion category** is a semisimple category with addition ($\oplus$) and multiplication ($\otimes$) with finitely many simple objects.

Isomorphism Classes:

- Each class is a group of objects that are isomorphic, or essentially equivalent.
- Modding out for categories.
- Irr$(\mathcal{C})$: Isomorphism classes of **simple** objects

Consider remainders when dividing integers by 5 ($\mathbb{Z}/5\mathbb{Z}$). The isomorphism classes here are $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$, where:

$$\overline{0} = \{..., -10, -5, 0, 5, 10, ...\}$$
$$\overline{1} = \{..., -9, -4, 1, 6, 11, ...\}$$
$$\overline{2} = \{..., -8, -3, 2, 7, 12, ...\}$$
$$\overline{3} = \{..., -7, -2, 3, 8, 13, ...\}$$
$$\overline{4} = \{..., -6, -1, 4, 9, 14, ...\}$$

Graded fusion categories and universal grading

$$S = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Let $\mathcal{C}$ be a fusion category and G a finite group. A *G-grading* on $\mathcal{C}$ is a decomposition of $\mathcal{C}$ into a direct sum of subcategories

$$\mathcal{C} = \sum_{g \in G} \mathcal{C}_g = \bigoplus_{g \in G} \mathcal{C}_g = \mathcal{C}_{g_1} \oplus \mathcal{C}_{g_2} \oplus \cdots \oplus \mathcal{C}_{g_k}$$

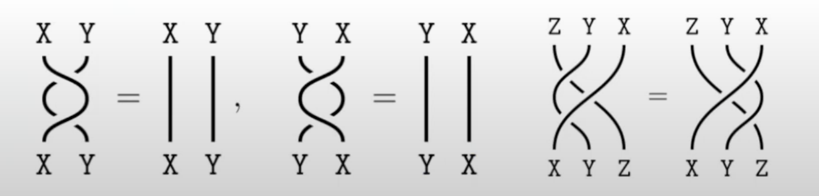
Summation is building up, **grading** is breaking down.

The **universal grading group** $U(\mathcal{C})$ is an 'omnipotent' group:

- It grades any category.
- Any G-grading of a category $\mathcal{C}$ is determined by the grading induced by $U(\mathcal{C})$ and the relationship between $U(\mathcal{C})$ and G.

Group Actions and Braidings

- Group action: Each element of G becomes a function on $\mathcal{C}$, so $g_k \mapsto f_{g_k}$, where $f_{g_k} : \mathcal{C} \to \mathcal{C}$
- Braiding: Twisting of strands, swapping order of objects under $\otimes$ such that some associativity conditions hold.
  - Map $c_{x,y} : x \otimes y \mapsto y \otimes x$



G-crossed braided fusion categories

A *G-crossed braided fusion category* is a fusion category $\mathcal{C}$ equipped with an action of G on $\mathcal{C}$, a grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, and G-braiding isomorphisms

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} g(Y) \otimes X \text{ for } g \in G,\ X \in \mathcal{C}_g,\ \text{and } Y \in \mathcal{C}.$$

Some additional structural conditions also must hold.

Exact factorization in fusion categories

Let $\mathcal{B}$ be a fusion category and let $\mathcal{A}, \mathcal{C}$ be fusion subcategories of $\mathcal{B}$.

$\mathcal{B}$ is an **exact factorization** of $\mathcal{A}$ and $\mathcal{C}$, denoted by $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$, if every simple object of $\mathcal{B}$ can be written uniquely as $A \otimes C$ with $A \in \mathsf{Irr}(\mathcal{A})$ and $C \in \mathsf{Irr}(\mathcal{C})$.

Our Results

Exact Factorizations for the Universal Grading

Let $\mathcal{B} = \mathcal{A} \cdot \mathcal{C}$ be an exact factorization of fusion categories. If $\mathcal{B}$ is $U(\mathcal{B})$-crossed braided, then $\mathcal{A}$ is $U(\mathcal{A})$-crossed braided and $\mathcal{C}$ is $U(\mathcal{C})$-crossed braided.

Fusion rings are skeletons of fusion categories.

Exact Factorizations for the Universal Grading (Rings)

If $R = A \cdot C$ is an exact factorization of fusing **rings** and R is a $U(R)$-crossed fusion ring, then A is a $U(A)$-crossed fusion ring and C is a $U(C)$-crossed fusion ring

Our Results (continued)

Let $\mathcal{B} = \mathcal{A} \cdot \mathcal{C}$ be an exact factorization. Define $H = \{g \in G, \mathcal{B}_g \cap \mathcal{A} \neq 0\}$ and $K = \{g \in G, \mathcal{B}_g \cap \mathcal{C} \neq 0\}$.

Factorizations for the General Case

$G = HK$ is a factorization.

Question

Is there an exact factorization of the trivial components of the gradings, $\mathcal{B}_e = \mathcal{A}_e \cdot \mathcal{C}_e$?

Corollary

If the above holds true, then $G = HK$ is an exact factorization.

Results

Deligne Product

Suppose $\mathcal{B} = \mathcal{A} \boxtimes \mathcal{C}$, where $\mathcal{A}$ is H-crossed braided and $\mathcal{C}$ is K-crossed braided. Then $\mathcal{B}$ is $H \times K$-crossed braided.

Crossed Product

$\mathcal{C} \rtimes G$ is G-crossed braided if and only if $\mathcal{C}$ is braided and ρ_g is isomorphic to $\mathrm{id}_{\mathcal{C}}$ for all $g \in G$.

Definition 5.2. Let $\mathcal{A}$ and $\mathcal{C}$ be fusion categories and G a finite group. Assume that G acts categorically by tensor autoequivalences on $\mathcal{A}$ and $\mathcal{C}$ has faithful G-grading. We define the *generalized semidirect product* $\mathcal{A} \rtimes \mathcal{C}$ as the fusion subcategory of $(\mathcal{A} \rtimes G) \boxtimes \mathcal{C}$ generated by the elements

$$(A\# \, |C|) \boxtimes C, \qquad A \in \mathrm{Irr}(\mathcal{A}), C \in \mathrm{Irr}(\mathcal{C}).$$

Remark 5.3. Notice that $\mathcal{A} \rtimes \mathcal{C}$ is as an abelian category $\mathcal{A} \boxtimes \mathcal{C}$. In this viewpoint we can denote the simple objects of $\mathcal{A} \rtimes \mathcal{C}$ by $A\#C$, $A \in \mathrm{Irr}(\mathcal{A})$, $C \in \mathrm{Irr}(\mathcal{C})$.

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